Gravitational Radiation from Point Masses in a Keplerian Orbit

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The gravitational radiation from two point masses going around each other under their mutual gravitational influence is calculated. Two different methods are outlined; one involves a multipole expansion of the radiation field, while the other uses the inertia tensor of the source. The calculations apply for arbitrary eccentricity of the relative orbit, but assume orbital velocities are small. The total rate, angular distribution, and polarization of the radiated energy are discussed.

I. INTRODUCTION

THE linearized version of Einstein's general theory of relativity is strikingly similar to classical electromagnetism. In particular, one might expect masses in arbitrary motion to radiate gravitational energy. The question has been raised,[1] however, whether the energy so calculated has any physical meaning. We shall not concern ourselves with this question here; we shall take the point of view that the analogy with electromagnetic theory is a correct one, and energy is actually radiated.

In Sec. II we outline briefly two methods which can be used to calculate rates of emission of gravitational energy from a system of masses on which no net external force acts. Only enough details are presented to enable them to be applied to other problems; derivations and proofs are omitted. In Sec. III these methods are applied to obtain the total rate of radiation by two point masses going around each other in the familiar Kepler ellipse. In Sec. IV we discuss the angular distribution and polarization of the radiation.

II. GENERAL METHODS

A. Inertia Tensor

If one linearizes the equations of general relativity, setting[2-4]

$$g_{\mu\nu}=\delta_{\mu\nu}+\kappa h_{\mu\nu}, \quad (|\kappa h_{\mu\nu}|\ll 1),$$

with $\kappa^2=32\pi G$, one obtains

$$\Box\bar{h}_{\mu\nu}=-\tfrac{1}{2}\kappa T_{\mu\nu}, \tag{1}$$

where

$$\bar{h}_{\mu\nu}=h_{\mu\nu}-\tfrac{1}{2}\delta_{\mu\nu}h_{\lambda\lambda},$$

and $T_{\mu\nu}$ is the total stress-momentum-energy tensor of the source, including the gravitational field stresses. The energy density in a plane wave

$$\bar{h}_{\mu\nu}=h_{\mu\nu}=ae_{\mu\nu}\cos(\omega t-\mathbf{k}\cdot\mathbf{x})$$

is

$$U=\tfrac{1}{2}c^2\omega^2a^2 \tag{2}$$

provided $e_{\mu\nu}$ is a unit polarization tensor, obeying the conditions

$$e_{\mu\nu}=e_{\nu\mu}, \quad e_{\mu\mu}=0, \quad k_\mu e_{\mu\nu}=0, \quad e_{\mu\nu}e_{\mu\nu}=1.$$

Just as in electromagnetic theory, we can work in a gauge in which $e_{\mu\nu}$ is spacelike and transverse; thus, a wave traveling in the z direction has two independent polarizations possible:

$$e_1=\frac{1}{\sqrt{2}}(\hat{x}\hat{x}-\hat{y}\hat{y}) \quad e_2=\frac{1}{\sqrt{2}}(\hat{x}\hat{y}+\hat{y}\hat{x}).$$

One can now solve (1) for the radiation from a system of masses undergoing arbitrary motions, and use (2) to obtain the power radiated. The result,[2] assuming source dimensions are small compared with the wavelength ("quadrupole approximation"), is that the power $dP/d\Omega$ radiated into solid angle Ω with polarization e_{ij} is

$$\frac{dP}{d\Omega}=\frac{G}{8\pi c^5}\left(\frac{d^3Q_{ij}}{dt^3}e_{ij}\right)^2, \tag{3}$$

where Q_{ij} is the tensor

$$Q_{ij}=\sum\nolimits_\alpha m_\alpha x_{\alpha i}x_{\alpha j}, \tag{4}$$

the sum running over all masses m_α in our system. It is to be noted that the result is independent of the kind of stresses present.

If one sums (3) over the two allowed polarizations, one obtains

$$\sum_{\text{pol}}\frac{dP}{d\Omega}=\frac{G}{8\pi c^5}\left[\frac{d^3Q_{ij}}{dt^3}\frac{d^3Q_{ij}}{dt^3}-2n_i\frac{d^3Q_{ij}}{dt^3}n_k\frac{d^3Q_{kj}}{dt^3}-\frac{1}{2}\left(\frac{d^3Q_{ii}}{dt^3}\right)^2 +\frac{1}{2}\left(n_in_j\frac{d^3Q_{ij}}{dt^3}\right)^2+\frac{d^3Q_{ii}}{dt^3}n_jn_k\frac{d^3Q_{jk}}{dt^3}\right], \tag{5}$$

where $\hat{n}$ is the unit vector in the direction of radiation. The total rate of radiation is obtained by integrating

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[1] See, for example, L. Infeld and J. Plebanski, *Motion and Relativity* (Pergamon Press Inc., New York, 1960).

[2] L. Landau and E. Lifshitz, *The Classical Theory of Fields* (Addison-Wesley Publishing Company, Inc., Reading, Massachusetts, 1959), Chap. 11.

[3] R. P. Feynman, lectures, California Institute of Technology (unpublished).

[4] Greek letters run from 1 to 4; $a_\mu b_\mu=a_4b_4-\mathbf{a}\cdot\mathbf{b}$. Roman letters run from 1 to 3; $a_ib_i=\mathbf{a}\cdot\mathbf{b}$. The Kronecker delta $\delta_{\mu\nu}$ is $+1$ for $\mu=\nu=4$, -1 for $\mu=\nu=1,2,3$. The d'Alembertian operator is $\Box=\nabla_\mu\nabla_\mu=\partial^2/\partial t^2-\nabla^2$. The phase of a plane wave is $k_\mu x_\mu=\omega t-\mathbf{k}\cdot\mathbf{x}$. G is the usual gravitational constant $\approx 6.67\times 10^{-8}$ cgs units.

(5) over all directions of emission; the result is

$$P=\frac{G}{5c^5}\left(\frac{d^3Q_{ij}}{dt^3}\frac{d^3Q_{ij}}{dt^3}-\frac{1}{3}\frac{d^3Q_{ii}}{dt^3}\frac{d^3Q_{jj}}{dt^3}\right). \qquad (6)$$

B. Multipole Expansion

The radiation $h_{\mu\nu}(\mathbf{x})$ can be decomposed into multipoles,[5] each with a definite total angular momentum (J) and z component of angular momentum (M). For a given J and M, there are two independent types of radiation, distinguished by their parity; we call them "electric" and "magnetic" to emphasize the analogy with electromagnetic theory.

We analyze the source and field into Fourier components, and treat each separately. If the source is

$$T_{\mu\nu}=\mathrm{Re}\hat{T}_{\mu\nu}e^{-i\omega t},$$

then the amplitudes of the electric and magnetic multipole radiation are

$$e_{JM}=-\frac{i\kappa\omega}{2}\int d^3x f_{JM}{}^e(\mathbf{x}):\hat{\mathrm{T}}(\mathbf{x}), \qquad (7)$$

$$m_{JM}=-\frac{i\kappa\omega}{2}\int d^3x f_{JM}{}^m(\mathbf{x}):\hat{\mathrm{T}}(\mathbf{x}), \qquad (8)$$

where $A:B$ means $A_{ij}B_{ij}$, and the $f_{JM}{}^{e,m}$ are given in reference 5. In the quadrupole approximation, the dominant type of radiation is "magnetic quadrupole"; in this limit, (8) with $J=2$ becomes

$$m_{2M}=\frac{i\kappa\omega^3}{10\sqrt{3}}\int d^3x\, r^2 Y_{2M}(\Omega)\hat{\rho}(\mathbf{x}), \qquad (9)$$

where

$$\rho=\mathrm{Re}\hat{\rho}e^{-i\omega t}$$

is the mass density in the source.

The total power radiated is given in terms of the multipole amplitudes (7), (8) by

$$P=\tfrac{1}{2}\textstyle\sum_{JM}[|e_{JM}|^2+|m_{JM}|^2]. \qquad (10)$$

III. TOTAL RADIATION

Let the masses m_1 and m_2 have coordinates $(d_1\cos\psi, d_1\sin\psi)$ and $(-d_2\cos\psi, -d_2\sin\psi)$ in the xy plane, as in Fig. 1. The origin will be taken to be the center of mass, so that

$$d_1=\left(\frac{m_2}{m_1+m_2}\right)d, \quad d_2=\left(\frac{m_1}{m_1+m_2}\right)d.$$

The simplest way to compute the power radiated is to use the method of Sec. II A, above. The nonvanishing

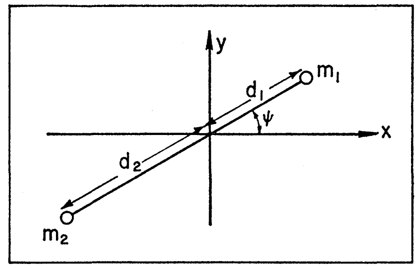

FIG. 1. Coordinate system used in calculation.

Q_{ij} are

$$Q_{xx}=\mu d^2\cos^2\psi,$$
$$Q_{yy}=\mu d^2\sin^2\psi,$$
$$Q_{xy}=Q_{yx}=\mu d^2\sin\psi\cos\psi,$$

where μ is the reduced mass $m_1m_2/(m_1+m_2)$.

For Kepler motion, the orbit equation is[6]

$$d=\frac{a(1-e^2)}{1+e\cos\psi}, \qquad (12)$$

while the angular velocity is given by

$$\dot{\psi}=\frac{[G(m_1+m_2)a(1-e^2)]^{1/2}}{d^2}. \qquad (13)$$

Using (12) and (13), it is straightforward to calculate the d^3Q_{ij}/dt^3; the results are

$$\frac{d^3Q_{xx}}{dt^3}=\beta(1+e\cos\psi)^2(2\sin2\psi+3e\sin\psi\cos^2\psi),$$
$$\frac{d^3Q_{yy}}{dt^3}=-\beta(1+e\cos\psi)^2\times[2\sin2\psi+e\sin\psi(1+3\cos^2\psi)], \qquad (14)$$
$$\frac{d^3Q_{xy}}{dt^3}=\frac{d^3Q_{yx}}{dt^3}=-\beta(1+e\cos\psi)^2\times[2\cos2\psi-e\cos\psi(1-3\cos^2\psi)],$$

where β is defined by

$$\beta^2=\frac{4G^3m_1{}^2m_2{}^2(m_1+m_2)}{a^5(1-e^2)^5}.$$

The total power radiated is now given by (6);

$$P=\frac{8}{15}\frac{G^4}{c^5}\frac{m_1{}^2m_2{}^2(m_1+m_2)}{a^5(1-e^2)^5}(1+e\cos\psi)^4\times[12(1+e\cos\psi)^2+e^2\sin^2\psi]. \qquad (15)$$

[5] J. Mathews, J. Soc. Ind. Appl. Math. **10**, 768 (1962). This expansion into multipoles is not to be confused with general multipole expansions usually given. See, for example, *Gravitation, an Introduction to Current Research*, edited by Louis Witten (John Wiley & Sons, Inc., New York, 1962), Chaps. 5 and 6.

[6] a is the semimajor axis and e the eccentricity of our ellipse. Note that we have chosen the x axis to be the direction of m_1 at its closest approach to m_2 (periastron).

In (15), ψ is, of course, the *retarded* position of the system. The *average* rate at which the system radiates energy is obtained by averaging (15) over one period of the elliptical motion; one obtains in this way

$$\langle P\rangle = \frac{32}{5}\frac{G^4}{c^5}\frac{m_1^2 m_2^2(m_1+m_2)}{a^5(1-e^2)^{7/2}}\left(1+\frac{73}{24}e^2+\frac{37}{96}e^4\right). \quad (16)$$

Thus, the average power equals the power radiated from a circular orbit of equal semimajor axis (or total energy) times an enhancement factor

$$f(e) = \frac{1+(73/24)e^2+(37/96)e^4}{(1-e^2)^{7/2}}. \quad (17)$$

Figure 2 shows $f(e)$ plotted against e. Note that $f(0.6)\sim 10$, $f(0.8)\sim 10^2$, $f(0.9)\sim 10^3$. The power radiated is a steeply rising function of the eccentricity e.

The same result follows from the method of Sec. II B, but the formalism is rather different. We must evaluate the m_{2M} of Eq. (9). In terms of the Q_{ij} defined by (4),

$$m_{2\pm2} = \frac{i\kappa\omega^3}{10\sqrt{3}}\left(\frac{15}{32\pi}\right)^{1/2}(Q_{xx}-Q_{yy}\pm 2iQ_{xy}),$$

$$m_{2\pm1} = 0,$$

$$m_{20} = \frac{-i\kappa\omega^3}{10\sqrt{3}}\left(\frac{5}{16\pi}\right)^{1/2}(Q_{xx}+Q_{yy}).$$

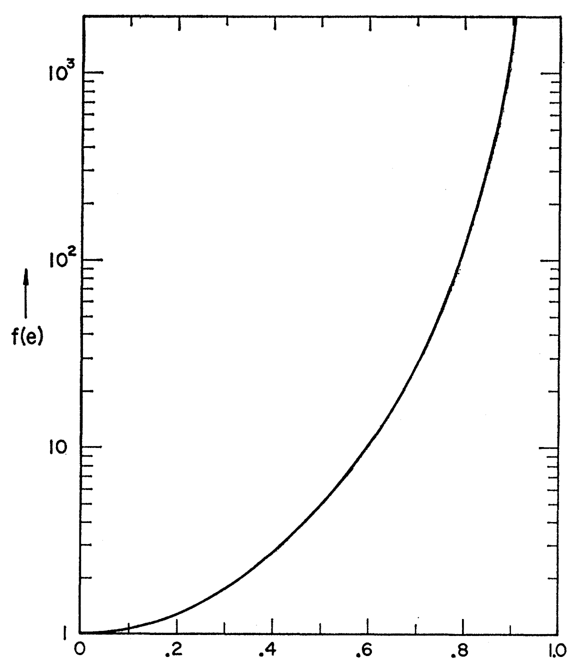

FIG. 2. "Enhancement factor" $f(e)$ plotted against e.

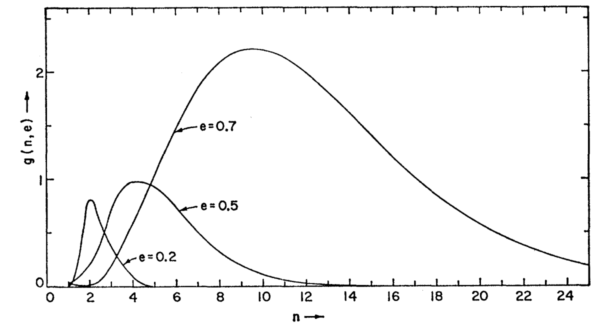

FIG. 3. $g(n,e)$, the relative power radiated into the nth harmonic for $e=0.2$, 0.5, and 0.7.

The Fourier analysis of Kepler motion is well known (to astronomers at least!), so we simply give the results. The components of frequency $n\omega_0$, where $\omega_0=[G(m_1+m_2)/a^3]^{1/2}$ is the average angular velocity, are

$$\begin{aligned} m_{2\pm2}(n) &= \frac{i\kappa\omega^3}{10\sqrt{3}}\left(\frac{15}{32\pi}\right)^{1/2}\mu a^2\frac{2}{n} \\ &\times\Big\{J_{n-2}(ne)-2eJ_{n-1}(ne)+\frac{2}{n}J_n(ne) \\ &+2eJ_{n+1}(ne)-J_{n+2}(ne) \\ &\mp(1-e^2)^{1/2}[J_{n-2}(ne)-2J_n(ne)+J_{n+2}(ne)]\Big\}, \\ m_{20}(n) &= \frac{i\kappa\omega^3}{10\sqrt{3}}\left(\frac{5}{16\pi}\right)^{1/2}\mu a^2\frac{4}{n^2}J_n(ne). \end{aligned} \quad (18)$$

The power radiated in the nth harmonic is, from (10) and (18),

$$P(n) = \frac{32}{5}\frac{G^4}{c^5}\frac{m_1^2 m_2^2(m_1+m_2)}{a^5}g(n,e), \quad (19)$$

where

$$\begin{aligned} g(n,e) = \frac{n^4}{32}\Big\{\Big[&J_{n-2}(ne)-2eJ_{n-1}(ne) \\ &+\frac{2}{n}J_n(ne)+2eJ_{n+1}(ne)-J_{n+2}(ne)\Big]^2 \\ &+(1-e^2)[J_{n-2}(ne)-2J_n(ne)+J_{n+2}(ne)]^2 \\ &+\frac{4}{3n^2}[J_n(ne)]^2\Big\}. \end{aligned} \quad (20)$$

In Fig. 3, we plot $g(n,e)$ against n for $e=0.2$, 0.5, and 0.7.

If (16) and (19) are to agree, we must have

$$\sum_{n=1}^{\infty} g(n,e) = f(e) = \frac{1+(73/24)e^2+(37/96)e^4}{(1-e^2)^{7/2}}.$$

This is verified in the Appendix.

That the radiation should depend so strongly on the eccentricity is not surprising. As with electrogmagnetic radiation, the power radiated increases for increasing accelerations. Thus, the bodies will radiate most at their closest approach, and for fixed energy the higher the eccentricity, the higher the power radiated will be. This also explains why the higher harmonics dominate the radiation for e near 1; Fourier components of large n must be present to give such a peaking of the radiation at one part of the path.

IV. ANGULAR DISTRIBUTIONS AND POLARIZATIONS

In this section we only use the method of Sec. II A, as it gives the answers directly without the need of summing over all harmonics.

Let us label the two polarizations

$$e_1=\frac{1}{\sqrt{2}}(\hat{\theta}\hat{\theta}-\hat{\phi}\hat{\phi}),\quad e_2=\frac{1}{\sqrt{2}}(\hat{\theta}\hat{\phi}+\hat{\phi}\hat{\theta}),\tag{21}$$

where θ and ϕ are conventional polar coordinates. We shall abbreviate the d^3Q_{ij}/dt^3 of (14) by A, B, C:

$$\frac{d^3Q_{xx}}{dt^3}=A,\quad \frac{d^3Q_{yy}}{dt^3}=B,\quad \frac{d^3Q_{xy}}{dt^3}=\frac{d^3Q_{yx}}{dt^3}=C.\tag{22}$$

The power radiated into polarization 1 is obtained by substituting (21) and (22) into (3); we omit the algebra and quote the result:

$$\begin{aligned}\frac{dP_1}{d\Omega}=\frac{G}{8\pi c^5}\{&\tfrac{1}{16}(3A^2+2AB+3B^2+4C^2)(1+\cos^4\theta)\\&-\tfrac{1}{8}(A^2+6AB+B^2-4C^2)\cos^2\theta\\&-\tfrac{1}{4}(A^2-B^2)(1-\cos^4\theta)\cos 2\phi\\&-\tfrac{1}{2}C(A+B)(1-\cos^4\theta)\sin 2\phi\\&+\tfrac{1}{16}[(A-B)^2-4C^2](1+\cos^2\theta)^2\cos 4\phi\\&+\tfrac{1}{4}C(A-B)(1+\cos^2\theta)^2\sin 4\phi\}.\end{aligned}\tag{23}$$

The result of averaging (23) over one period of the motion is

$$\begin{aligned}\left\langle\frac{dP_1}{d\Omega}\right\rangle=\frac{1}{\pi}\frac{G^4}{c^5}\frac{m_1^2m_2^2(m_1+m_2)}{a^5(1-e^2)^{7/2}}\Bigg[&\left(\frac{1}{2}+\frac{99}{64}e^2+\frac{51}{256}e^4\right)\\&\times(1+\cos^4\theta)+\left(1+\frac{95}{32}e^2+\frac{47}{128}e^4\right)\cos^2\theta\\&+\left(\frac{13}{32}e^2+\frac{1}{16}e^4\right)(1-\cos^4\theta)\cos 2\phi\\&-\frac{25}{512}e^4(1+\cos^2\theta)^2\cos 4\phi\Bigg].\end{aligned}$$

The corresponding results for polarization 2 of (21) are

$$\begin{aligned}\frac{dP_2}{d\Omega}=\frac{G}{8\pi c^5}\Bigg\{&\frac{1}{4}[4C^2+(A-B)^2]\cos^2\theta\\&+\frac{1}{4}[4C^2-(A-B)^2]\cos^2\theta\cos 4\phi\\&+C(B-A)\cos^2\theta\sin 4\phi\Bigg\},\end{aligned}\tag{24}$$

$$\begin{aligned}\left\langle\frac{dP_2}{d\Omega}\right\rangle=\frac{1}{\pi}\frac{G^4}{c^5}\frac{m_1^2m_2^2(m_1+m_2)}{a^5(1-e^2)^{7/2}}&\\\times\Bigg[\left(2+\frac{97}{16}e^2+\frac{49}{64}e^4\right)\cos^2\theta&\\+\frac{25}{128}e^4\cos^2\theta\cos 4\phi\Bigg].&\end{aligned}$$

The total power radiated into both polarizations may be obtained either by adding (23) and (24), or by using (5) directly. The result is

$$\begin{aligned}\frac{dP}{d\Omega}=\frac{G}{8\pi c^5}\{&(\tfrac{1}{16})(3A^2+2AB+3B^2+4C^2)(1+\cos^4\theta)\\&+\tfrac{1}{8}(A^2-10AB+B^2+12C^2)\cos^2\theta\\&+\tfrac{1}{4}(B^2-A^2)(1-\cos^4\theta)\cos 2\phi\\&-\tfrac{1}{2}C(A+B)(1-\cos^4\theta)\sin 2\phi\\&+\tfrac{1}{16}[(A-B)^2-4C^2]\sin^4\theta\cos 4\phi\\&+\tfrac{1}{4}C(A-B)\sin^4\theta\sin 4\phi\}.\end{aligned}\tag{25}$$

The average of (25) over the orbit is

$$\begin{aligned}\left\langle\frac{dP}{d\Omega}\right\rangle=\frac{1}{\pi}\frac{G^4}{c^5}\frac{m_1^2m_2^2(m_1+m_2)}{a^5(1-e^2)^{7/2}}&\\\times\{[\tfrac{1}{2}+(99/64)e^2+(51/256)e^4](1+\cos^4\theta)&\\+[3+(289/32)e^2+(145/128)e^4]\cos^2\theta&\\+(13/32)e^2+(1/16)e^4](1-\cos^4\theta)\cos 2\phi&\\-(25/512)e^4\sin^4\theta\cos 4\phi\}.&\end{aligned}$$

The basic results of this section, Eqs. (23), (24), and (25), are quite complicated. The quantities A, B, and C are given by (22) and (14) as functions of ψ, the retarded orientation of the line joining the mass points. We may extract some rather simple results from our formulas, however.

For example, in the case of circular motion ($e=0$), the formulas become

$$\frac{dP_1}{d\Omega}=\frac{1}{\pi}\frac{G^4}{c^5}\frac{m_1^2m_2^2(m_1+m_2)}{a^5}(1+\cos^2\theta)^2\sin^2 2(\phi-\psi),$$

$$\frac{dP_2}{d\Omega}=\frac{4}{\pi}\frac{G^4}{c^5}\frac{m_1^2m_2^2(m_1+m_2)}{a^5}\cos^2\theta\cos^2 2(\phi-\psi),$$

$$\frac{dP}{d\Omega}=\frac{1}{\pi}\frac{G^4}{c^5}\frac{m_1^2m_2^2(m_1+m_2)}{a^5}[4\cos^2\theta+\sin^4\theta\sin^2 2(\phi-\psi)].$$

The averages over the orbit are now quite trivially done:

$$\left\langle\frac{dP_1}{d\Omega}\right\rangle=\frac{1}{2\pi}\frac{G^4}{c^5}\frac{m_1^2m_2^2(m_1+m_2)}{a^5}(1+\cos^2\theta)^2,$$

$$\left\langle\frac{dP_2}{d\Omega}\right\rangle=\frac{2}{\pi}\frac{G^4}{c^5}\frac{m_1^2m_2^2(m_1+m_2)}{a^5}\cos^2\theta,$$

$$\left\langle\frac{dP}{d\Omega}\right\rangle=\frac{1}{2\pi}\frac{G^4}{c^5}\frac{m_1^2m_2^2(m_1+m_2)}{a^5}(1+6\cos^2\theta+\cos^4\theta).$$

Another aspect of Eqs. (23)–(25) is that the total power may be obtained by integrating over solid angle, and the result for the total power should agree with (15). Carrying out the integration over all directions, we obtain

$$\begin{aligned}P_1&=(G/120c^5)(11A^2-6AB+11B^2+28C^2),\\ P_2&=(G/120c^5)(5A^2-10AB+5B^2+20C^2),\\ P&=(2G/15c^5)(A^2-AB+B^2+3C^2).\end{aligned}\qquad(26)$$

The corresponding averages over the elliptical orbit are

$$\langle P_1\rangle=\frac{32}{5}\frac{G}{c^5}\frac{m_1^2m_2^2(m_1+m_2)}{a^5(1-e^2)^{7/2}}\left(\frac{7}{12}+\frac{683}{384}e^2+\frac{347}{1536}e^4\right),\qquad(27)$$

$$\langle P_2\rangle=\frac{32}{5}\frac{G}{c^5}\frac{m_1^2m_2^2(m_1+m_2)}{a^5(1-e^2)^{7/2}}\left(\frac{5}{12}+\frac{485}{384}e^2+\frac{245}{1536}e^4\right).\qquad(28)$$

It is straightforward to verify that (26), with A, B, C given by (22) and (14), agrees with our previous result (15), and that the sum of (27) and (28) is just the value (16) for $\langle P\rangle$ given earlier.

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APPENDIX

We now show that the sum over all harmonics n of $g(n,e)$ is the same as $f(e)$, where $g(n,e)$ is defined by (20) and $f(e)$ is given by (17).

We first reduce the right-hand side of Eq. (20) to terms containing only $[J_n(ne)]^2$, $J_n'(ne)J_n(ne)$, and $[J_n'(ne)]^2$, by use of the recurrence relations and Bessel's equation. Prime denotes differentiation with respect to the argument. This gives

$$\begin{aligned}g(n,e)=\frac{n^4}{32}\Bigg\{&\frac{J_n^2}{n^2}\left(2-\frac{4}{e^2}\right)^2+J_n'^2\left(\frac{4}{e}-4e\right)^2+\frac{2J_nJ_n'}{n}\\ &\times\left(2-\frac{4}{e^2}\right)\left(\frac{4}{e}-4e\right)+(1-e^2)J_n^2\left(\frac{4}{e^2}-4\right)^2\\ &+(1-e^2)\frac{J_n'^2}{n^2}\left(\frac{4}{e^2}\right)^2-\frac{2J_nJ_n'}{n}(1-e^2)\left(\frac{4}{e}\right)\\ &\times\left(\frac{4}{e^2}-4\right)+\frac{4}{3n^2}J_n^2\Bigg\}.\end{aligned}\qquad(A1)$$

A solution of the equation $M=E-e\sin E$ for $E(M,e)$ is given by the Fourier expansion

$$E(M,e)=M+2\sum_{n=1}^{\infty}\frac{\sin(nM)}{n}J_n(ne).$$

If we differentiate (A2) successively with respect to e, we can form series with terms such as $\sin(nM)J_n'$, $\sin(nM)nJ_n$, $\sin(nM)n^2J_n'$, and $\sin(nM)n^3J_n$. We have made use of Bessel's equation to eliminate terms with a higher than first derivative of J_n. If we multiply two such series together, say,

$$\left[\frac{\partial^2E}{\partial e^2}+\frac{1}{e}\frac{\partial E}{\partial e}\right]^2=\frac{4(1-e^2)}{e^4}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\sin(nM)\times\sin(mM)nmJ_n(ne)J_m(me),$$

and integrate both sides with respect to M from 0 to 2π, we get on the right-hand side

$$\frac{4(1-e^2)^2\pi}{e^4}\sum_{n=1}^{\infty}n^2J_n^2(ne),$$

which is one of the expressions needed to sum (A1). The integral on the left-hand side is straightforward. The formulas obtained in this manner which are necessary to sum (A1) are

$$\sum_{n=0}^{\infty}n^2J_n^2(ne)=\frac{e^2}{4(1-e^2)^{7/2}}\left(1+\frac{e^2}{4}\right),$$

$$\sum_{n=0}^{\infty}n^3J_n'(ne)J_n(ne)=\frac{e}{4(1-e^2)^{9/2}}\left(1+3e^2+\frac{3}{8}e^4\right),$$

$$\sum_{n=0}^{\infty}n^4[J_n'(ne)]^2=\frac{1}{4(1-e^2)^{11/2}}\times\left(1+\frac{39}{4}e^2+\frac{79}{8}e^4+\frac{45}{64}e^6\right),\qquad(A3)$$

$$\sum_{n=0}^{\infty} n^2[J_n'(ne)]^2=\frac{1}{4(1-e^2)^{5/2}}\left(1+\frac{3e^2}{4}\right),$$

$$\sum_{n=0}^{\infty} n^4 J_n^2(ne)=\frac{e^2}{4(1-e^2)^{13/2}}\left(1+\frac{37}{4}e^2+\frac{59}{8}e^4+\frac{27}{64}e^6\right).$$

Substitution of (A3) into the sum of the reduced series (Al) yields

$$\sum_{n=1}^{\infty} g(n,e)=\frac{1+\frac{73}{24}e^2+\frac{37}{96}e^4}{(1-e^2)^{7/2}},$$

which is the same as $f(e)$ as calculated in (17).

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Quasiparticles and the Born Series*

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Perturbation theory always works in nonrelativistic scattering theory, unless composite particles are present. By "composite particle" is meant a bound state or resonance, or one that would exist for an interaction of opposite sign; in fact, this provides a precise definition of resonances. It follows that if fictitious elementary particles (quasiparticles) are first introduced to take the place of all composite particles, then perturbation theory can always be used. There are several ways of accomplishing this, one of which corresponds to the N/D method. In order to prove these results it is necessary to make a detailed study of the eigenvalues of the scattering kernel, and as a by-product we obtain new proofs of the applicability of the Fredholm theorems to scattering theory, of the convergence of the Born series at high energy, of the Bargmann-Schwinger theorem on the number of bound states, of the Pais-Jost theorem on the identity of the Jost function with the Fredholm determinant, and of Levinson's theorem. We also give explicit formulas for binding energies and phase shifts in potential theory, using first-order perturbation theory after insertion of a single quasiparticle; these formulas work well for the lowest bound state and the S-wave scattering length of the Yukawa potential, and give precisely 13.6 eV for the hydrogen atom binding energy.

I. INTRODUCTION

THIS is the second of a series of papers, in which we hope to develop a practicable method of calculating strong interaction processes.

In our first paper[1] it was proven that any given nonrelativistic Hamiltonian H can be rewritten to introduce fictitious elementary particles (quasiparticles) which did not appear in H. The new Hamiltonian $\mathbf{H}$ yields precisely the same physical predictions as H, provided that when we put the quasiparticles into the unperturbed part, we also modify the interaction term according to certain rules. These matters are reviewed in Sec. II.

We also remarked in A that such quasiparticles can be introduced very freely, without any reference to physically real particles, and also without any point. But their introduction can be the crucial step in practical calculations, for such calculations can always be done by perturbation methods unless composite particles are present. If we introduce a quasiparticle corresponding to each composite particle, then we get a new (but physically equivalent) theory in which there are no composites, but only real and fictitious elementary particles, so that perturbation theory works. What actually happens is that the modification of the Hamiltonian forced upon us by the introduction of a quasiparticle weakens the original interaction enough to remove the divergence of the Born series associated with the corresponding composite particle.[2] Seen in this way, the strength of a given coupling should never make us despair of applying perturbation theory; a very strong interaction merely gives rise to many composite particles, and, hence, forces us to introduce a large number of quasiparticles before we start using the Born series.

I believe that this approach will make perturbation theory universally applicable, even to the full relativistic series of Feynman diagrams.[3] The purpose of this paper is to demonstrate that this conjecture is, indeed, correct within the limited proving ground of nonrelativistic two-body scattering theory.

It is shown in Sec. III that the Born series will diverge if and only if there are composite particles present, and

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[1] S. Weinberg, Phys. Rev. **130**, 776 (1963); this article will be referred to as A.

[2] A more general approach to the problem of obtaining a convergent perturbation series has been suggested by M. Rotenberg (to be published). Our approach seems to correspond to his if the operator he calls "$J-1$" is chosen to be separable; otherwise the quasiparticle interpretation is inapplicable.

[3] Some preliminary steps in this direction are reported by S. Weinberg, in *Proceedings of the 1962 Annual International Conference on High-Energy Physics at CERN*, edited by J. Prentki (CERN, Geneva, 1962), p. 683.